

Examples on Lagrange's Equations, for n-degree of freedom systems:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} = Q_j \quad ; j = 1, 2, 3, \dots, n$$

Or as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} = Q_j \quad ; j = 1, 2, 3, \dots, n$$

These equations represent a system of n -differential equations, one corresponding to each of the n generalized coordinates.

q_j : generalized displacement for the j^{th} -mass

$\dot{q}_j$: generalized velocity for the j^{th} -mass

Q_j : Force applied to the j^{th} -mass

T : Kinetic energy of the system

V : Potential energy of the system

R : Rayleigh's dissipation function

L : Lagrangian $L = T - V$

To understand this method study the following examples carefully

Chapter 6

6.39 Coordinates of the bob are $(x + l \cos \theta, l \sin \theta)$

$T = \text{kinetic energy} = \text{kinetic energy of slider} + \text{kinetic energy of bob}$

$$= \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} m \left[\left\{ \frac{d}{dt} (x + l \cos \theta) \right\}^2 + \left\{ \frac{d}{dt} (l \sin \theta) \right\}^2 \right]$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 (\sin^2 \theta + \cos^2 \theta) - \frac{1}{2} m (2 \dot{x} l \sin \theta \dot{\theta})$$

$$= m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 - m \dot{x} \dot{\theta} l \sin \theta \simeq m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 \text{ for small } \theta.$$

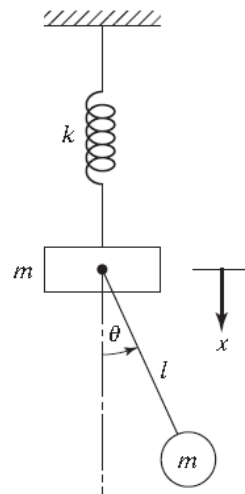
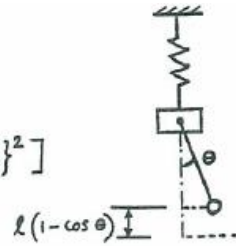


FIGURE 6.36

$V = \text{potential energy} = \text{potential energy of spring} + \text{potential energy of bob}$

$$= \frac{1}{2} k x^2 + m g l (1 - \cos \theta)$$

(Note: Potential energy of slider need not be considered if $x=0$ corresponds to static equilibrium position)

Since $\cos \theta \simeq 1 - \frac{1}{2} \theta^2$, $V = \frac{1}{2} k x^2 + \frac{1}{2} m g l \theta^2$

As there are no external forces, Lagrange's equations become

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 ; \quad j = 1, 2$$

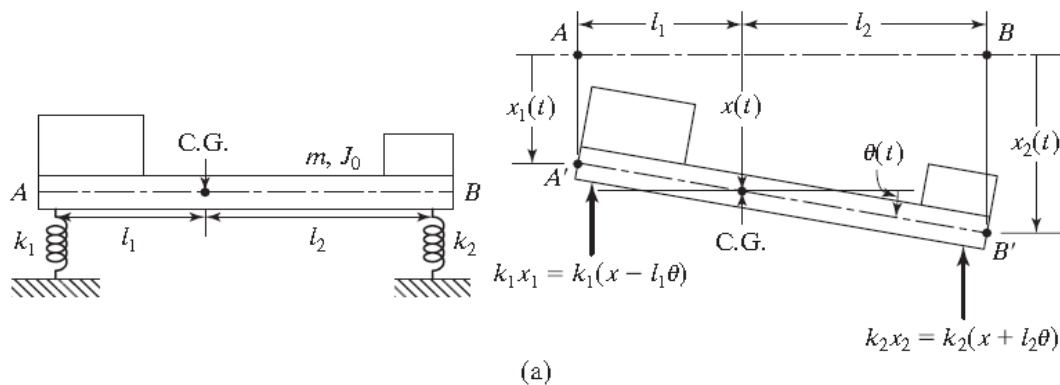
Here $q_1 = x$ and $q_2 = \theta$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m \dot{x}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x}, \quad \frac{\partial V}{\partial x} = k x$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = m l^2 \dot{\theta}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = m g l \theta$$

Lagrange's equations become

$$m \ddot{x} + k x = 0 ; \quad m l^2 \ddot{\theta} + m g l \theta = 0 \quad \text{or} \quad l \ddot{\theta} + g \theta = 0$$



6.40

(1) With x_1 and x_2 as generalized coordinates:

Since $x_1 = x - l_1 \theta$ and $x_2 = x + l_2 \theta$,

$$x = \left(\frac{x_1 l_2 + x_2 l_1}{l_1 + l_2} \right) \quad \text{and} \quad \theta = \left(\frac{x_2 - x_1}{l_1 + l_2} \right)$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m \left(\frac{\dot{x}_1 l_2 + \dot{x}_2 l_1}{l_1 + l_2} \right)^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}_2 - \dot{x}_1}{l_1 + l_2} \right)^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = \frac{m}{2(l_1 + l_2)^2} (2 l_2^2 \dot{x}_1 + 2 l_1 l_2 \dot{x}_2) + \frac{J_0}{2(l_1 + l_2)^2} (2 \dot{x}_1 - 2 \dot{x}_2),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = \frac{m}{(l_1 + l_2)^2} (l_2^2 \ddot{x}_1 + l_1 l_2 \ddot{x}_2) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_1 - \ddot{x}_2)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = \frac{m}{2(l_1 + l_2)^2} (2 l_1^2 \dot{x}_2 + 2 l_1 l_2 \dot{x}_1) + \frac{J_0}{2(l_1 + l_2)^2} (2 \dot{x}_2 - 2 \dot{x}_1),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = \frac{m}{(l_1 + l_2)^2} (l_1^2 \ddot{x}_2 + l_1 l_2 \ddot{x}_1) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_2 - \ddot{x}_1)$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

$$\frac{\partial V}{\partial x_1} = k_1 x_1, \quad \frac{\partial V}{\partial x_2} = k_2 x_2$$

Lagrange's equations, Eq. (6.41), give

$$\frac{m}{(l_1 + l_2)^2} (l_2^2 \ddot{x}_1 + l_1 l_2 \ddot{x}_2) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_1 - \ddot{x}_2) + k_1 x_1 = 0$$

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$$\frac{m}{(l_1 + l_2)^2} (l_1^2 \ddot{x}_2 + l_1 l_2 \ddot{x}_1) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_2 - \ddot{x}_1) + k_2 x_2 = 0$$

$$\text{i.e.} \quad \ddot{x}_1 (m l_2^2 + J_0) + \ddot{x}_2 (m l_1 l_2 - J_0) + x_1 (l_1 + l_2)^2 k_1 = 0$$

$$\ddot{x}_1 (m l_1 l_2 - J_0) + \ddot{x}_2 (m l_1^2 + J_0) + x_2 (l_1 + l_2)^2 k_2 = 0$$

(2) With x and θ as generalized coordinates:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 (x - l_1 \theta)^2 + \frac{1}{2} k_2 (x + l_2 \theta)^2$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m \dot{x}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x}, \quad \frac{\partial V}{\partial x} = k_1 (x - l_1 \theta) + k_2 (x + l_2 \theta)$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = J_0 \ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = -k_1 l_1 (x - l_1 \theta) + k_2 l_2 (x + l_2 \theta)$$

Lagrange's equations give

$$m \ddot{x} + k_1 (x - l_1 \theta) + k_2 (x + l_2 \theta) = 0$$

$$J_0 \ddot{\theta} - k_1 l_1 (x - l_1 \theta) + k_2 l_2 (x + l_2 \theta) = 0$$

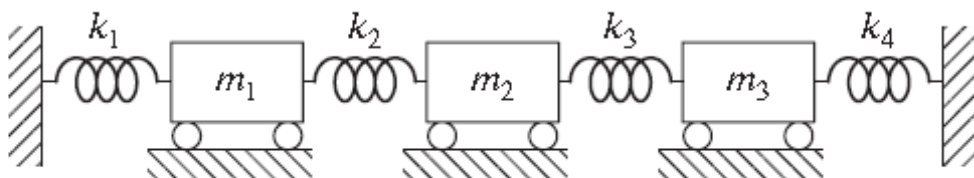


FIGURE 6.29

6.41

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2 + \frac{1}{2} k_4 x_3^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1, \quad \frac{\partial V}{\partial x_1} = k_1 x_1 - k_2 (x_2 - x_1)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2, \quad \frac{\partial V}{\partial x_2} = k_2 (x_2 - x_1) - k_3 (x_3 - x_2)$$

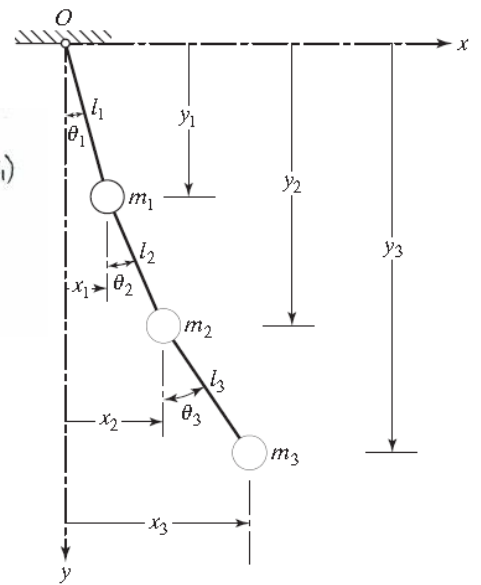
$$\frac{\partial T}{\partial x_3} = 0, \quad \frac{\partial T}{\partial \dot{x}_3} = m_3 \dot{x}_3, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = m_3 \ddot{x}_3, \quad \frac{\partial V}{\partial x_3} = k_3 (x_3 - x_2) + k_4 x_3$$

Lagrange's equations give

$$m_1 \ddot{x}_1 + x_1 (k_1 + k_2) - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 - k_2 x_1 + x_2 (k_2 + k_3) - k_3 x_3 = 0$$

$$m_3 \ddot{x}_3 - k_3 x_2 + x_3 (k_3 + k_4) = 0$$



6.42

Using θ_1 , θ_2 and θ_3 as generalized coordinates, we find

$$T \approx \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2 + \frac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2 + \frac{1}{2} m_3 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3)^2 \dots (E_1)$$

Let reference position correspond to $\theta_1 = \theta_2 = \theta_3 = 0$.

Vertical movement of m_1 is

$$l_1 (1 - \cos \theta_1) \approx l_1 \left\{ 1 - \left(1 - \frac{\theta_1^2}{2} \right) \right\} = \frac{1}{2} l_1 \theta_1^2$$

$$\text{vertical movement of } m_2 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2$$

$$\text{vertical movement of } m_3 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2 + \frac{1}{2} l_3 \theta_3^2$$

$$V = \frac{m_1 g l_1 \theta_1^2}{2} + \frac{m_2 g}{2} (l_1 \theta_1^2 + l_2 \theta_2^2) + \frac{m_3 g}{2} (l_1 \theta_1^2 + l_2 \theta_2^2 + l_3 \theta_3^2) \dots (E_2)$$

$$\frac{\partial T}{\partial \theta_1} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) = (m_1 + m_2 + m_3) l_1^2 \ddot{\theta}_1 + (m_2 + m_3) l_1 l_2 \ddot{\theta}_2 + m_3 l_1 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2 + m_3) g l_1 \theta_1$$

$$\frac{\partial T}{\partial \theta_2} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_2} = m_2 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) = (m_2 + m_3) l_1 l_2 \ddot{\theta}_1 + (m_2 + m_3) l_2^2 \ddot{\theta}_2 + m_3 l_2 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \theta_2 + m_3 g l_2 \theta_2 = (m_2 + m_3) g l_2 \theta_2$$

$$\frac{\partial T}{\partial \theta_3} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_3} = m_3 l_3 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_3} \right) = m_3 l_3 (l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + l_3 \ddot{\theta}_3), \quad \frac{\partial V}{\partial \theta_3} = m_3 g l_3 \theta_3$$

Lagrange's equations give the equations of motion

$$\begin{bmatrix} (m_1 + m_2 + m_3) l_1^2 & (m_2 + m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2 + m_3) l_1 l_2 & (m_2 + m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} (m_1 + m_2 + m_3) g l_1 & 0 & 0 \\ 0 & (m_2 + m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l_3 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \dots (E_3)$$

6.43

(a) Let generalized coordinates be $q_1 = x$ and $q_2 = \theta$ Displacement of mass M is $x + l\theta$.Kinetic energy of airplane is $T = \frac{1}{2} M_0 \dot{x}^2 + 2 \left\{ \frac{1}{2} M (\dot{x} + l\dot{\theta})^2 \right\}$ Potential energy is $V = 2 \left(\frac{1}{2} k_t \theta^2 \right)$ Lagrange's equations are $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$; $i = 1, 2$ Here $\frac{\partial T}{\partial x} = 0$, $\frac{\partial T}{\partial \dot{x}} = M_0 \dot{x} + 2M(\dot{x} + l\dot{\theta})$, $\frac{\partial V}{\partial x} = 0$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = 2Ml(\dot{x} + l\dot{\theta}), \quad \frac{\partial V}{\partial \theta} = 2k_t\theta, \quad Q_i = 0$$

Hence Lagrange's equations become

$$\left. \begin{aligned} M_0 \ddot{x} + 2M(\ddot{x} + l\ddot{\theta}) &= 0 \\ 2Ml(\ddot{x} + l\ddot{\theta}) + 2k_t\theta &= 0 \end{aligned} \right\}$$

(b) If $x(t) = X \cos(\omega t + \phi)$ and $\theta(t) = \Theta \cos(\omega t + \phi)$ the equations of motion become

$$\left. \begin{aligned} (-M_0\omega^2 - 2M\omega^2)X - 2Ml\omega^2\Theta &= 0 \\ -2Ml\omega^2 X - 2Ml^2\omega^2\Theta + 2k_t\Theta &= 0 \end{aligned} \right\}$$

which yields the frequency equation:

$$\begin{vmatrix} M_0\omega^2 + 2M\omega^2 & 2Ml\omega^2 \\ 2Ml\omega^2 & 2Ml^2\omega^2 - 2k_t \end{vmatrix} = 0$$

$$\text{i.e., } \omega^2 [2M_0Ml^2\omega^2 - 2M_0k_t - 4Mk_t] = 0$$

$$\text{i.e., } \omega^2 = 0; \quad \omega^2 = \left(\frac{2M_0k_t + 4Mk_t}{2M_0Ml^2} \right)$$

Mode shapes: Eq. (E₂) gives

$$\frac{\Theta}{X} = \frac{2Ml\omega^2}{-2Ml^2\omega^2 + 2k_t}$$

$$\text{For } \omega_1 = 0; \quad \frac{\Theta}{X} \Big|_{\omega_1} = \frac{0}{2k_t} = 0 \Rightarrow \Theta = 0$$

This corresponds to rigid body translation in x (vertical) direction.

$$\begin{aligned} \text{For } \omega_2; \quad \frac{\Theta}{X} \Big|_{\omega_2} &= \frac{2Ml \left(\frac{2M_0k_t + 4Mk_t}{2M_0Ml^2} \right)}{-2Ml^2 \left(\frac{2M_0k_t + 4Mk_t}{2M_0Ml^2} \right) + 2k_t} \\ &= - \left(\frac{M_0}{2Ml} + \frac{1}{l} \right) \end{aligned}$$

(c) For $\omega_2 > 4\pi$ rad/sec,

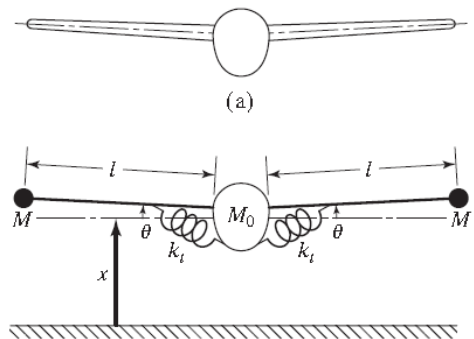
$$\left(\frac{2M_0k_t + 4Mk_t}{2M_0Ml^2} \right) > 16\pi^2 \quad (E_3)$$

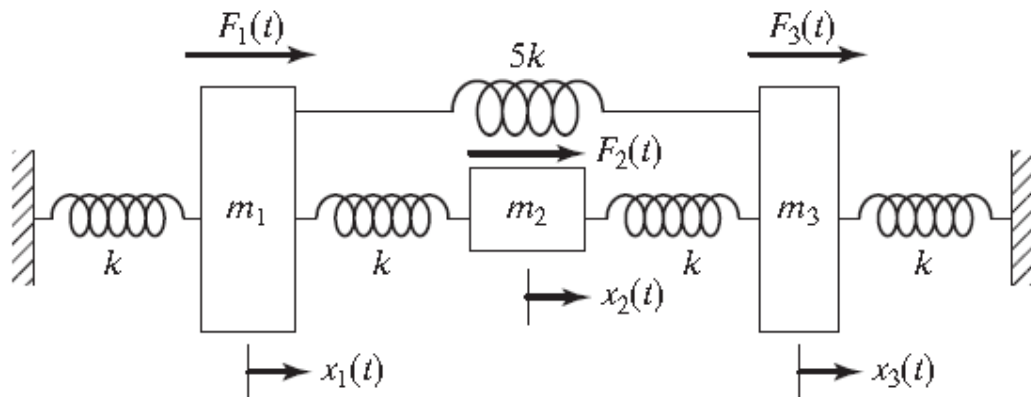
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When $M_0 = 1000$ kg, $M = 500$ kg and $l = 6$ m, inequality(E₃) becomes

$$\frac{2000k_t + 2000k_t}{(1 \times 10^6)(36)} > 16\pi^2$$

$$\text{i.e., } k_t > 1.4212 \times 10^6 \text{ N-m/rad.}$$





6.44

Generalized coordinates: x_1, x_2, x_3 :

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} (5k) (x_3 - x_1)^2$$

$$Q_i = F_i ; i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_1} = 7 k x_1 - k x_2 - 5 k x_3$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_i} \right) = m_i \ddot{x}_i ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_2} = -k x_1 + 2 k x_2 - k x_3$$

$$\frac{\partial T}{\partial x_i} = 0 ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_3} = -5 k x_1 - k x_2 + 7 k x_3$$

Lagrange's equations yield the equations of motion:

$$m_1 \ddot{x}_1 + 7 k x_1 - k x_2 - 5 k x_3 = F_1(t)$$

$$m_2 \ddot{x}_2 - k x_1 + 2 k x_2 - k x_3 = F_2(t)$$

$$m_3 \ddot{x}_3 - 5 k x_1 - k x_2 + 7 k x_3 = F_3(t)$$

6.45

Generalized coordinates: θ , x_1 , x_2 :

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

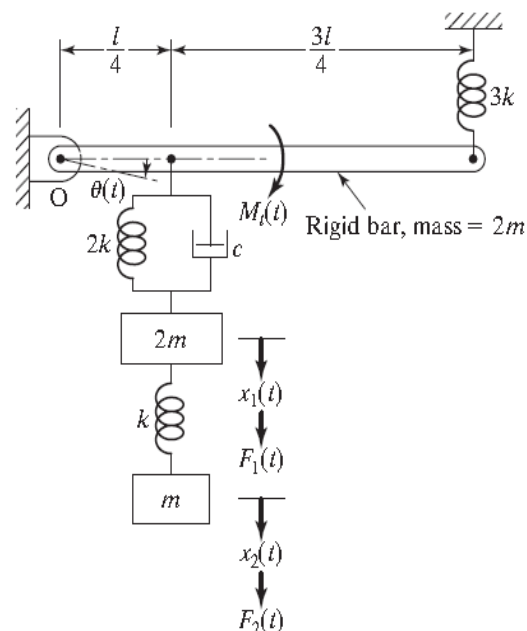
$$\text{where } J_0 = \frac{1}{3} (2m) \ell^2 = \frac{2}{3} m \ell^2.$$

$$V = \frac{1}{2} (2k) (x_1 - \frac{\ell}{4} \theta)^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} (3k) (\theta \ell)^2$$

$$R = \frac{1}{2} c (\dot{x}_1 - \frac{\ell}{4} \dot{\theta})^2$$

$$Q_\theta = M_t(t) ; Q_1 = F_1(t) ; Q_2 = F_2(t)$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} ; \frac{\partial T}{\partial \dot{x}_1} = 2 m \dot{x}_1 ; \frac{\partial T}{\partial \dot{x}_2} = m \dot{x}_2$$



$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = J_0 \ddot{\theta} ; \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = 2 m \ddot{x}_1 ; \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m \ddot{x}_2$$

$$\frac{\partial T}{\partial \theta} = 0 ; \frac{\partial T}{\partial x_i} = 0 ; i = 1, 2$$

$$\frac{\partial R}{\partial \dot{\theta}} = -c \frac{\ell}{4} (\dot{x}_1 - \dot{\theta} \frac{\ell}{4}) ; \frac{\partial R}{\partial \dot{x}_1} = c (\dot{x}_1 - \dot{\theta} \frac{\ell}{4}) ; \frac{\partial R}{\partial \dot{x}_2} = 0$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{25}{8} k \ell^2 \right) \theta - \frac{1}{2} k \ell x_1$$

$$\frac{\partial V}{\partial x_1} = -\frac{1}{2} k \ell \theta + 3 k x_1 - k x_2$$

$$\frac{\partial V}{\partial x_2} = -k x_1 + k x_2$$

Lagrange's equations, Eqs. (6.118), yield the equations of motion as:

$$\begin{aligned} \frac{2}{3} m \ell^2 \ddot{\theta} + \frac{1}{16} c \ell^2 \dot{\theta} - \frac{1}{4} c \ell \dot{x}_1 + \frac{25}{8} k \ell^2 \theta - \frac{1}{2} k \ell x_1 &= M_t(t) \\ 2 m \ddot{x}_1 - \frac{1}{4} c \ell \dot{\theta} + c \dot{x}_1 - \frac{1}{2} k \ell \theta + 3 k x_1 - k x_2 &= F_1(t) \\ m \ddot{x}_2 - k x_1 + k x_2 &= F_2(t) \end{aligned}$$

6.46

Generalized coordinates: x_i ; $i = 1, 2, 3$:

$$T = \frac{1}{2} J_G \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_G^2 + \frac{1}{2} (5m) \dot{x}_2^2$$

$$= \frac{1}{2} \left(\frac{25}{6} m \ell^2 \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right)^2 + \frac{1}{2} (2m) \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)^2 + \frac{1}{2} (5m) \dot{x}_2^2$$

where the subscript G denotes the mass center of the bar with

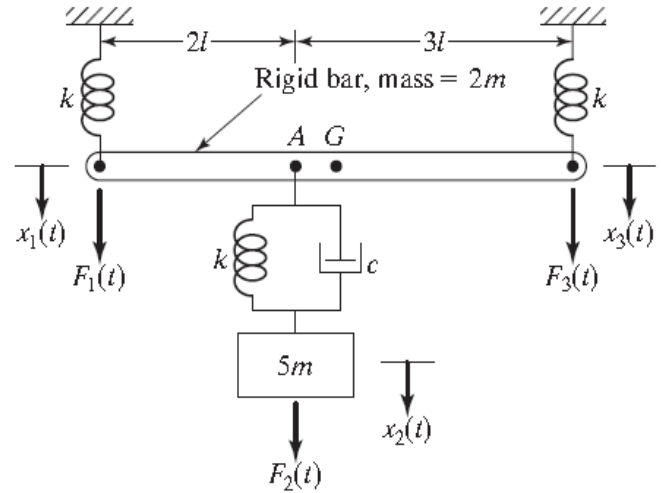
$$J_G = \frac{1}{12} (2m) (5 \ell)^2 = \frac{25}{6} m \ell^2 ; \quad \theta = \frac{x_1 - x_3}{5 \ell} ; \quad x_G = \frac{x_1 + x_3}{2}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} k (x_A - x_2)^2$$

$$= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} k \left(\frac{3x_1 + 2x_3}{5} - x_2 \right)^2$$

since $x_A = x_1 - \frac{2}{5} (x_1 - x_3) = \frac{3x_1 + 2x_3}{5}$

$$R = \frac{1}{2} c (\dot{x}_A - \dot{x}_2)^2 = \frac{1}{2} c \left(\frac{3\dot{x}_1 + 2\dot{x}_3}{5} - \dot{x}_2 \right)^2$$



$$Q_i(t) = F_i(t) ; \quad i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_1} = \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{5 \ell} \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right) + (2m) \frac{1}{2} \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{25 \ell^2} \right) (\dot{x}_1 - \ddot{x}_3) + \frac{m}{2} (\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial \dot{x}_2} = (5m) \dot{x}_2 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = (5m) \ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = - \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{5 \ell} \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right) + (2m) \frac{1}{2} \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = - \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{25 \ell^2} \right) (\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2} (\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial \dot{x}_i} = 0 ; \quad i = 1, 2, 3$$

$$\frac{\partial R}{\partial \dot{x}_1} = \frac{3}{5} c \left(\frac{3\dot{x}_1 + 2\dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_1} = k x_1 + \frac{3}{5} k \left(\frac{3x_1 + 2x_3}{5} - x_2 \right)$$

$$\frac{\partial R}{\partial \dot{x}_3} = \frac{2}{5} c \left(\frac{3\dot{x}_1 + 2\dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_2} = -k \left(\frac{3x_1 + 2x_3}{5} - x_2 \right)$$

$$\frac{\partial R}{\partial \dot{x}_2} = -c \left(\frac{3\dot{x}_1 + 2\dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_3} = k x_3 + \frac{2}{5} k \left(\frac{3x_1 + 2x_3}{5} - x_2 \right)$$

Application of Lagrange's equations gives the equations of motion as:

$$\frac{2}{3} m \ddot{x}_1 + \frac{1}{3} m \ddot{x}_3 + \frac{9}{25} c \dot{x}_1 - \frac{3}{5} c \dot{x}_2 + \frac{6}{25} c \dot{x}_3 + \frac{34}{25} k x_1 - \frac{3}{5} k x_2 + \frac{6}{25} k x_3 = F_1(t)$$

$$5 m \ddot{x}_2 - \frac{3}{5} c \dot{x}_1 + c \dot{x}_2 - \frac{2}{5} c \dot{x}_3 - \frac{3}{5} k x_1 + k x_2 - \frac{2}{5} k x_3 = F_2(t)$$

$$\frac{1}{3} m \ddot{x}_1 + \frac{2}{3} m \ddot{x}_3 + \frac{6}{25} c \dot{x}_1 - \frac{2}{5} c \dot{x}_2 + \frac{4}{25} c \dot{x}_3 + \frac{6}{25} k x_1 - \frac{2}{5} k x_2 + \frac{29}{25} k x_3 = F_3(t)$$

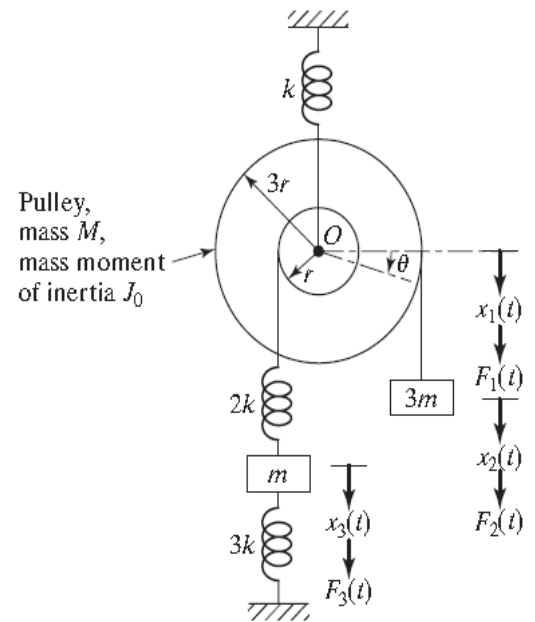
6.47

$$T = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (3m) \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} (2k) (x_1 - x_3 - r \theta)^2 + \frac{1}{2} (3k) x_3^2$$

$$Q_i = F_i ; i = 1, 2, 3$$

Noting that $\theta = \left(\frac{x_2 - x_1}{3r} \right)$, we can express



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$$\frac{\partial T}{\partial \dot{x}_1} = M \dot{x}_1 + J_0 \left(-\frac{1}{3r} \right) \left(\frac{\dot{x}_2 - \dot{x}_1}{3r} \right) ; \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = M \ddot{x}_1 - \frac{J_0}{3r} \left(\frac{\ddot{x}_2 - \ddot{x}_1}{3r} \right)$$

$$\frac{\partial T}{\partial \dot{x}_2} = J_0 \left(\frac{1}{3r} \right) \left(\frac{\dot{x}_2 - \dot{x}_1}{3r} \right) + 3m \dot{x}_2 ; \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = \frac{J_0}{9r^2} (\ddot{x}_2 - \ddot{x}_1) + 3m \ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = m \dot{x}_3 ; \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = m \ddot{x}_3$$

$$\frac{\partial V}{\partial x_1} = k x_1 + (2k) \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right)$$

$$\frac{\partial V}{\partial x_2} = -\frac{1}{3} (2k) \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right)$$

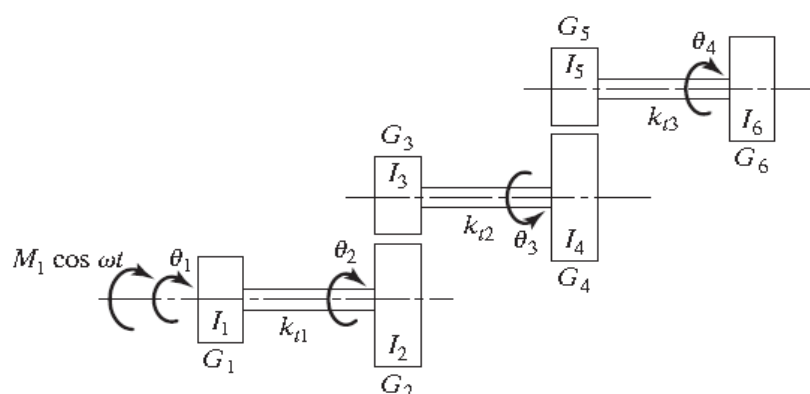
$$\frac{\partial V}{\partial x_3} = -(2k) \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right) + (3k) x_3$$

Application of Lagrange's equations give the equations of motion:

$$\left(M + \frac{J_0}{9r^2} \right) \ddot{x}_1 - \frac{J_0}{9r^2} \ddot{x}_2 + \frac{41}{9} k x_1 - \frac{8}{9} k x_2 - \frac{8}{3} k x_3 = F_1(t)$$

$$-\frac{J_0}{9r^2} \ddot{x}_1 + \left(3m + \frac{J_0}{9r^2} \right) \ddot{x}_2 - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 = F_2(t)$$

$$m \ddot{x}_3 - \frac{8}{3} k x_1 + \frac{2}{3} k x_2 + 5k x_3 = F_3(t)$$



Number of teeth on gear $G_i = n_i$ ($i = 1$ to 6)
 Mass moment of inertia of gear $G_i = I_i$ ($i = 1$ to 6)

6.48

Generalized coordinates: θ_i ; $i = 1, 2, 3, 4$:

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \dot{\theta}_2^2 + \frac{1}{2} \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \dot{\theta}_3^2 + \frac{1}{2} I_6 \dot{\theta}_4^2$$

$$V = \frac{1}{2} k_{t1} (\theta_2 - \theta_1)^2 + \frac{1}{2} k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right)^2 + \frac{1}{2} k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right)^2$$

$$Q_1 = M_1 \cos \omega t$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = I_1 \dot{\theta}_1 ; \quad \frac{\partial T}{\partial \dot{\theta}_2} = \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \dot{\theta}_3 ; \quad \frac{\partial T}{\partial \dot{\theta}_4} = I_6 \dot{\theta}_4$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) = I_1 \ddot{\theta}_1 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) = \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \ddot{\theta}_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_3} \right) = \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \ddot{\theta}_3 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_4} \right) = I_6 \ddot{\theta}_4$$

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$$\frac{\partial V}{\partial \theta_1} = -k_{t1} (\theta_2 - \theta_1)$$

$$\frac{\partial V}{\partial \theta_2} = k_{t1} (\theta_2 - \theta_1) - k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) \frac{n_2}{n_3}$$

$$\frac{\partial V}{\partial \theta_3} = k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) - k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) \frac{n_4}{n_5}$$

$$\frac{\partial V}{\partial \theta_4} = k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right)$$

Equations of motion (from Lagrange's equations):

$$I_1 \ddot{\theta}_1 - k_{t1} (\theta_2 - \theta_1) = M_1 \cos \omega t$$

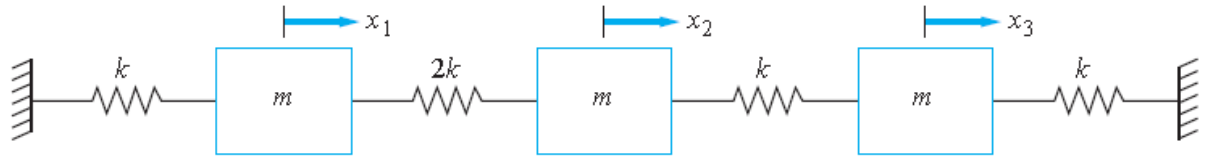
$$\left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \ddot{\theta}_2 + k_{t1} (\theta_2 - \theta_1) - k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) \frac{n_2}{n_3} = 0$$

$$\left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \ddot{\theta}_3 + k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) - k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) \frac{n_4}{n_5} = 0$$

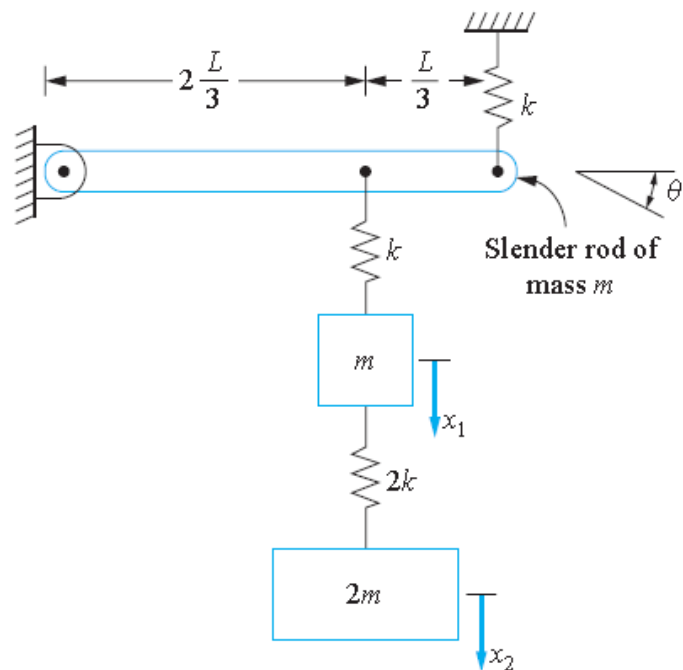
$$I_6 \ddot{\theta}_4 + k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) = 0$$

Using Lagrange's method, derive the equations of motion for the following systems:

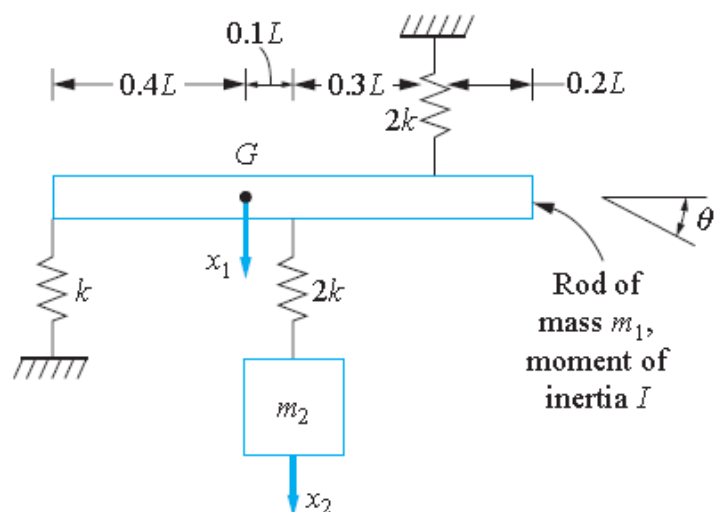
A)



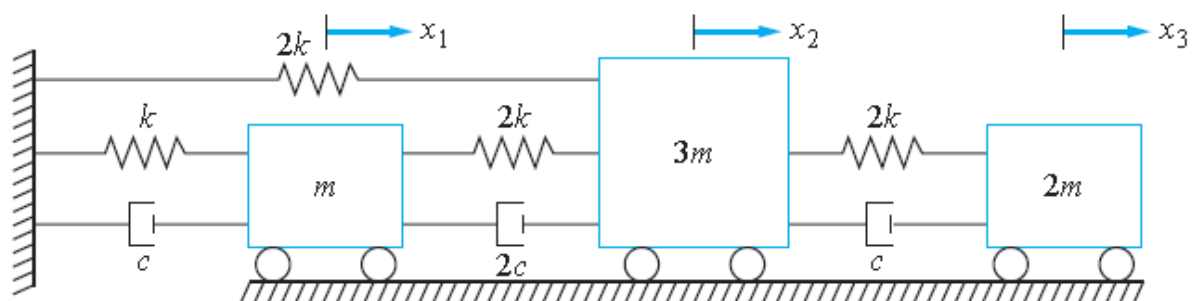
B)



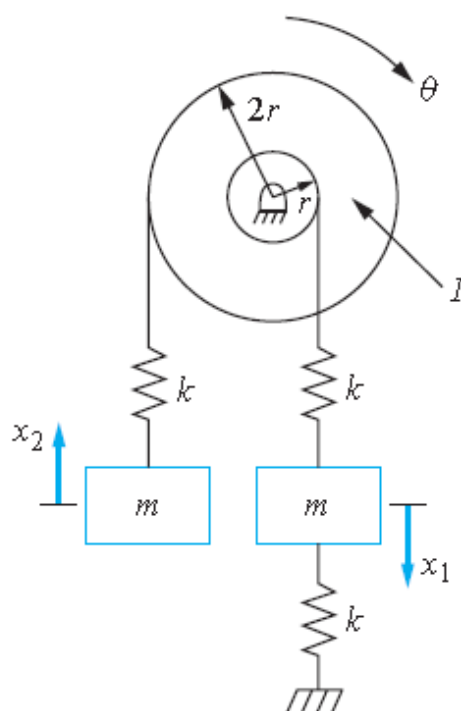
C)



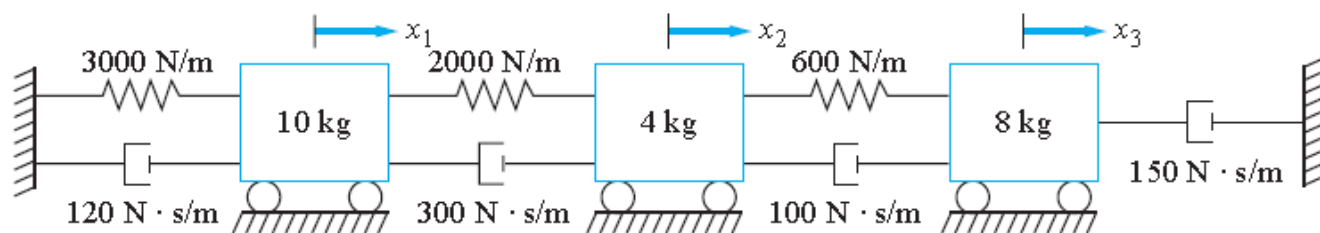
D)



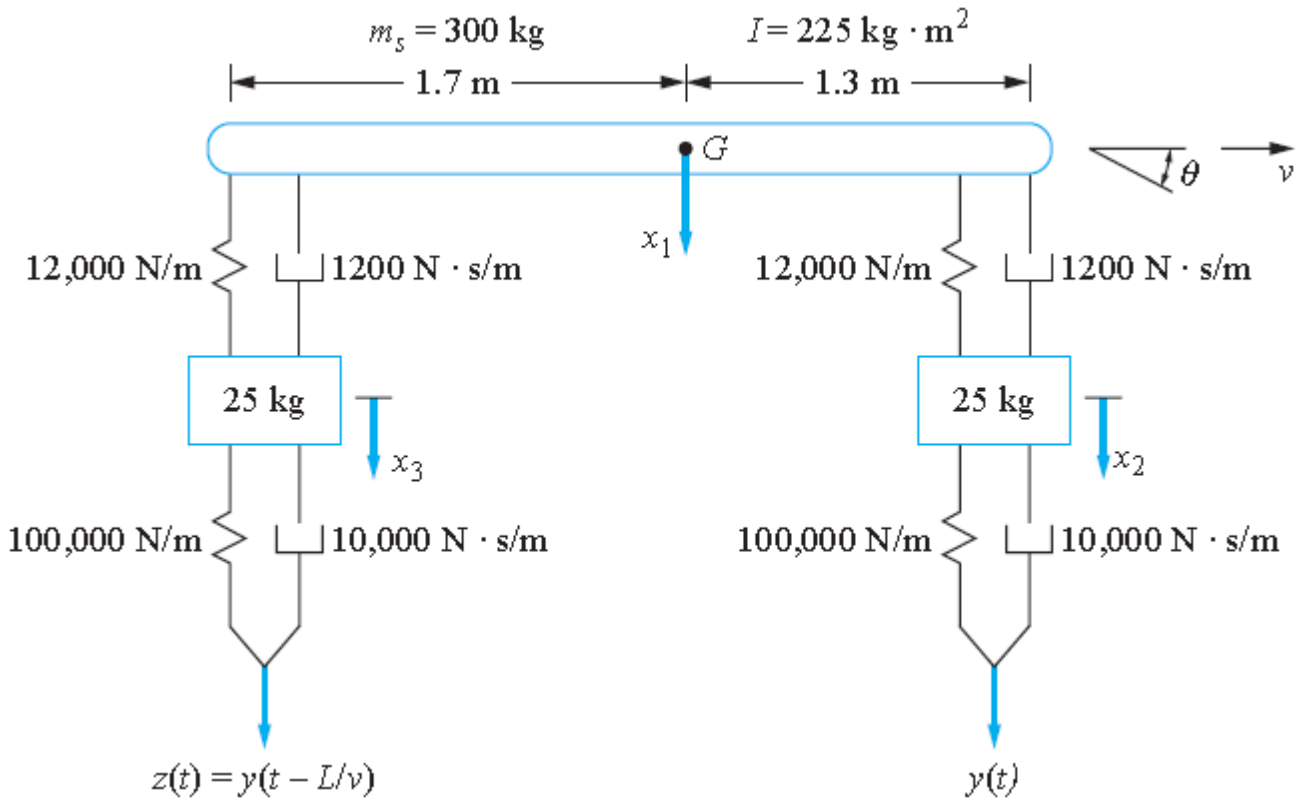
E)



F)



G) For the car model:



Where

kinetic energy of the car at an arbitrary instant is

$$T = \frac{1}{2} m_s \dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_a \dot{x}_2^2 + \frac{1}{2} m_a \dot{x}_3^2 \quad (\text{a})$$

The potential energy of the car at an arbitrary instant is

$$V = \frac{1}{2} k [x_2 - (x_1 + a\theta)]^2 + \frac{1}{2} k [x_3 - [x_1 - (L - a)\theta]]^2 + \frac{1}{2} k_t (y - x_2)^2 + \frac{1}{2} k_t (z - x_3)^2 \quad (\text{b})$$

And the Rayleigh's dissipation function R is given as:

$$\begin{aligned} & -\frac{1}{2} c [\dot{x}_2 - (\dot{x}_1 + a\dot{\theta})]^2 - \frac{1}{2} c [\dot{x}_3 - [\dot{x}_1 - (L - a)\dot{\theta}]]^2 \\ & - \frac{1}{2} c_t (\dot{y} - \dot{x}_2)^2 + \frac{1}{2} c_t (\dot{z} - \dot{x}_3)^2 \end{aligned}$$